

# Black holes thermodynamics to all orders in the Planck length in extra dimensions

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**Abstract.** We investigate the effects to all orders in the Planck length, from a generalized uncertainty principle (GUP), on the thermodynamic parameters of radiating Schwarzschild black holes in a scenario with large extra dimensions. We show that black holes in this framework are hotter, have less degrees of freedom and decay faster compared to black holes in the Hawking picture and in the framework with GUP to leading order in the Planck length. Particularly, we show that the final stage of the evaporation process is a black hole remnant with zero entropy, zero heat capacity and non zero finite temperature. We finally compare our results with the ones obtained in the standard Hawking picture and with the generalized uncertainty principle to leading order in the Planck length.

*Keywords:* black holes, extra dimensions, quantum gravity phenomenology, generalized uncertainty principle

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## 1. Introduction

The avenue of large extra dimension models (LED) offers new exciting ways to solve the hierarchy problem and to study low scale quantum gravity effects. The model of Arkani-Hamed, Dimopoulos and Dvali (ADD) [1, 2, 3] used  $n$  new large spacelike dimensions without curvature, and gravity is the only force which propagates in the full volume of the space-time (the bulk). Hence the gravitational force in the four-dimensional world (the brane) appears weak compared to the other forces which do not propagate in the extra dimensions. An alternative model proposed by Randall and Sundrum (RS) used warped extra dimension with a non factorisable geometry [4, 5]. In RS models gravity is diluted by the strong curvature of the extra dimension. Within these models the Planck scale is lowered to values soon accessible of order of a TeV. Among the predicted effects, the experimental production of black holes (BHs) at particle colliders such as the Large Hadronic Collider (LHC) [6] and the muon collider [7], is one of the most exciting possibility which has received a great amount of interest [8]-[33]. The newly formed BH is expected to decay instantaneously on collider detector time scales (typically of order  $10^{-26}$ s for LHC). At this scale the evaporation of BH is expected to end leaving up a possible Planck size black hole remnant (BHR).

Recently, a great interest has been devoted to the study of effects of generalized uncertainty principles (GUPs) and modified dispersion relations (MDRs) on various quantum gravity problems [34]-[38]. The GUPs and MDRs originates from several studies in string theory approach to quantum gravity [39]-[42], loop quantum gravity [43], noncommutative space-time algebra [44, 45] and black holes gedanken experiments [46, 47]. Actually, GUPs and MDRs are considered as common features of any promising candidate to a quantum theory of gravity.

In four dimensions, the consequences of GUPs and/or MDRs on BHs thermodynamics have been considered intensively in the recent literature on the subject [48]-[52], notably it has been shown that GUP prevents black holes from complete evaporation exactly like the standard Heisenberg principle prevents the hydrogen atom from total collapse [53]. Then, at the final stage of the Hawking evaporation process of a black hole, a inert black hole remnant (BHR) continue to exist with zero entropy, zero heat capacity and a finite non zero temperature. The inert character of the BHR, besides gravitational interactions, renders this object a serious candidate to explain the origin of dark matter [54, 55]. A particular attention has been also devoted to the computation of the entropy and to the sub-leading logarithmic correction [56]-[64]. The phenomenological properties of Black holes in the framework of the ADD model with GUP have been recently also studied [26, 65].

Until now all of the work has been done with GUP in the leading order in the fundamental length. However, a version of the GUP with higher orders in the Planck length induces quantitative corrections to the entropy and then influences the Hawking evaporation of the black hole [66]. Then, the ultimate quantum nature of the physics at the Planck scale would be best described in the framework of a GUP containing

the gravitational effects to all orders in the Planck length . In this framework, the corrections to BH thermodynamic parameters may have important consequences on BHs production at particle colliders.

In this paper we discuss the effects, to all orders in the Planck length, that a GUP may have on thermodynamic parameters of the Schwarzschild BH in the ADD model. The organization of this work is as follows. In section 2, we introduce a deformed position and momentum operators algebra leading to the GUP and examine quantum properties of this algebra. In section 3, the GUP-corrected thermodynamic parameters are computed and the departures from the standard semiclassical description shown. In section 4, we investigate the Hawking evaporation process and calculate exactly the evaporation rate and the decay time. We compare our results with the ones obtained in the context of the semiclassical description and with the GUP to the leading order in the Planck length. Our conclusions are summarized in the last section.

## 2. All orders corrections of GUP

One of the most interesting consequences of all promising quantum gravity candidates is the existence of a minimal observable length on the order of the Planck length. Actually, part of the work in quantum gravity phenomenology has been tackled with effective models based on MDRs and/or GUPs and containing the minimal length as a natural UV cut-off. The relation between these approaches has been recently clarified and established [68].

The idea of a minimal length can be modeled in terms of a quantized space-time and goes back to the early days of quantum field theory [69] (see also [70] – [73] ). An alternative approach is to consider deformations to the standard Heisenberg algebra [45], which lead to generalized uncertainty principles showing the existence of the minimal length. In this section we follow the latter approach and exploit a result recently obtained in the context of canonical noncommutative field theory in the coherent states representation [74] and field theory on non-anticommutative superspace [75, 76]. Indeed, it has been shown that the Feynman propagator displays an exponential UV cut-off of the form  $\exp(-\eta p^2)$ , where the parameter  $\eta$  is related to the minimal length. This framework has been further applied, in series of papers [77], to the black hole evaporation process.

At the quantum mechanical level, the UV finiteness of the Feynman propagator can be also captured by a non linear relation,  $k = f(p)$ , between the wave vector and the momentum of the particle [68]. This relation must be invertible and has to fulfil the following requirements:

- (i) For energies much smaller than the cut-off the usual dispersion relation is recovered.
- (ii) The wave vector is bounded by the cut-off.

In this picture, the usual commutator between the commuting position and

momentum operators is generalized to

$$[X, P] = i\hbar \frac{\partial p}{\partial k} \Leftrightarrow \Delta X \Delta P \geq \frac{\hbar}{2} \left| \left\langle \frac{\partial p}{\partial k} \right\rangle \right|, \quad (1)$$

and the momentum measure  $d^n p$  is deformed as  $d^n p \prod_i \frac{\partial k_i}{\partial p_j}$ . In the following, we will restrict ourselves to the isotropic case in one space-like dimension. Following [74, 76] and setting  $\eta = \frac{\alpha L_{Pl}^2}{\hbar^2}$  we have

$$\frac{\partial p}{\partial k} = \hbar \exp \left( \frac{\alpha L_{Pl}^2}{\hbar^2} p^2 \right), \quad (2)$$

where  $\alpha$  is a dimensionless constant of order one.

From Eq.(2) we obtain the dispersion relation

$$k(p) = \frac{\sqrt{\pi}}{2\sqrt{\alpha} L_{Pl}} \operatorname{erf} \left( \frac{\sqrt{\alpha} L_{Pl}}{\hbar} p \right), \quad (3)$$

from which we have the following minimum Compton wavelength

$$\lambda_0 = 4\sqrt{\pi\alpha} L_{Pl}. \quad (4)$$

We note that a dispersion relation similar to the one given by Eq.(3) has been used recently to investigate the effect of the minimal length on the running gauge couplings [78]. In the context of trans-Planckian physics, modified dispersion relations have been also used to study the spectrum of the cosmological fluctuations. A particular class of MDRs frequently used in the literature [79, 80] is the well known Unruh dispersion relations given by  $k(p) = \tanh^{1/\gamma}(p^\gamma)$ , with  $\gamma$  being some positive integer [81].

Let us show that the above results can be obtained from the following momentum space representation of the position and momentum operators

$$X = i\hbar \exp \left( \frac{\alpha L_{Pl}^2}{\hbar^2} P^2 \right) \partial_p \quad P = p. \quad (5)$$

The corrections to the standard Heisenberg algebra become effective in the so-called quantum regime where the momentum and length scales are of the order of the Planck mass  $M_{Pl}$  and the Planck length  $L_{Pl}$  respectively.

The hermiticity condition of the position operator implies modified completeness relation and modified scalar product given by

$$\int dp e^{-\frac{\alpha L_{Pl}^2}{\hbar^2} p^2} |p\rangle \langle p| = 1 \quad (6)$$

$$\langle p | p' \rangle = e^{\frac{\alpha L_{Pl}^2}{\hbar^2} p^2} \delta(p - p'). \quad (7)$$

From Eq.(6), we observe that we have reproduced the Gaussian damping factor in the Feynman propagator [74, 76].

The algebra defined by Eq. (5) leads to the following generalized commutator and generalized uncertainty principle (GUP)

$$[X, P] = i\hbar \exp \left( \frac{\alpha L_{Pl}^2}{\hbar^2} P^2 \right), \quad (\delta X)(\delta P) \geq \frac{\hbar}{2} \left\langle \exp \left( \frac{\alpha L_{Pl}^2}{\hbar^2} P^2 \right) \right\rangle. \quad (8)$$

In order to investigate the quantum mechanical implications of this deformed algebra, we solve the relation (8) for  $(\delta P)$  with the equality. Using the property  $\langle P^{2n} \rangle \geq \langle P^2 \rangle$  and  $(\delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2$ , the generalized uncertainty relation is written as

$$(\delta X)(\delta P) = \frac{\hbar}{2} \exp \left( \frac{\alpha L_{Pl}^2}{\hbar^2} ((\delta P)^2 + \langle P \rangle^2) \right). \quad (9)$$

Taking the square of this expression we obtain

$$W(u) e^{W(u)} = u, \quad (10)$$

where we have set  $W(u) = -2 \frac{\alpha L_{Pl}^2}{\hbar^2} (\delta P)^2$  and  $u = -\frac{\alpha L_{Pl}^2}{2(\delta X)^2} e^{-2 \frac{\alpha L_{Pl}^2}{\hbar^2} \langle P \rangle^2}$ .

The equation given by Eq.(10) is exactly the definition of the Lambert function [82], which is a multi-valued function. Its different branches,  $W_k(u)$ , are labeled by the integer  $k = 0, \pm 1, \pm 2, \dots$ . When  $u$  is a real number Eq.(10) have two real solutions for  $0 \geq u \geq -\frac{1}{e}$ , denoted by  $W_0(u)$  and  $W_{-1}(u)$ , or it can have only one real solution for  $u \geq 0$ , namely  $W_0(u)$ . For  $-\infty < u < -\frac{1}{e}$ , Eq.(10) have no real solutions.

Finally, the momentum uncertainty is given by

$$(\delta P) = \frac{\hbar}{\sqrt{2\alpha} L_{Pl}} \left( -W \left( -\frac{\alpha L_{Pl}^2}{2(\delta X)^2} e^{-2 \frac{\alpha L_{Pl}^2}{\hbar^2} \langle P \rangle^2} \right) \right)^{1/2}. \quad (11)$$

From the argument of the Lambert function we have the following condition

$$\frac{\alpha L_{Pl}^2 e^{\frac{2\alpha L_{Pl}^2}{\hbar^2} \langle P \rangle^2}}{2(\delta X)^2} \leq \frac{1}{e}, \quad (12)$$

which leads to a minimal uncertainty in position given by

$$(\delta X)_{\min} = \sqrt{\frac{e\alpha}{2}} L_{Pl} e^{\frac{\alpha L_{Pl}^2}{\hbar^2} \langle P \rangle^2}. \quad (13)$$

The absolutely smallest uncertainty in position or minimal length is obtained for physical states for which we have  $\langle P \rangle = 0$  and  $(\delta P) = \hbar / (\sqrt{2\alpha} L_{Pl})$ , and is given by

$$(\delta X)_0 = \sqrt{\frac{\alpha e}{2}} L_{Pl}. \quad (14)$$

In terms of the minimal length the momentum uncertainty becomes

$$(\delta P) = \frac{\hbar \sqrt{e}}{2(\delta X)_0} \left( -W \left( -\frac{1}{e} \left( \frac{(\delta X)_0}{(\delta X)} \right)^2 \right) \right)^{1/2}. \quad (15)$$

Here we observe that  $\frac{1}{e} \frac{(\delta X)_0}{(\delta X)} < 1$  is a small parameter by virtue of the GUP, and perturbative expansions to all orders in the Planck length can be safely performed.

Indeed, a series expansion of Eq.(15) gives the corrections to the standard Heisenberg principle

$$\delta P \simeq \frac{\hbar}{2(\delta X)} \left( 1 + \frac{1}{2e} \left( \frac{(\delta X)_0}{(\delta X)} \right)^2 + \frac{5}{8e^2} \left( \frac{(\delta X)_0}{(\delta X)} \right)^4 + \frac{49}{48e^3} \left( \frac{(\delta X)_0}{(\delta X)} \right)^6 + \dots \right). \quad (16)$$

This expression of  $(\delta P)$  containing only odd powers of  $(\delta X)$  is consistent with a recent analysis in which string theory and loop quantum gravity, considered as the most serious candidates for a theory of quantum gravity, put severe constraints on the possible forms of GUPs and MDRs [35].

Let us now recall the form of the GUP to leading order in the Planck length. This GUP is given by

$$(\delta X)(\delta P) \geq \frac{\hbar}{2} \left( 1 + \frac{\alpha L_{Pl}^2}{\hbar^2} (\delta P)^2 \right). \quad (17)$$

A simple calculation leads to the following minimal length

$$(\delta X)_0 = \sqrt{\alpha} L_{Pl}, \quad (18)$$

which is of order of the Planck length. However, the form of the GUP to leading order in the Planck length leads to a modified dispersion relation which does not fulfill the second requirement listed above [68]. In our case, It is easy to show that the wave vector given by (3) is bounded by the cut-off  $1/L_{Pl}$ . This observation may significantly influence the thermodynamics parameters and the evaporation process of small BHs.

In the following sections we use the form of the GUP given by Eq.(8) and investigate the thermodynamics of the Schwarzschild BH. We use units  $\hbar = c = k_B = 1$ .

### 3. Black hole thermodynamics with GUP

Black holes in higher dimensional space-times have been studied by Myers and Perry [83]. They considered the form of the gravitational background around an uncharged  $d$ -dimensional BH. In the non-rotating case, corresponding to a  $d$ -dimensional spherically symmetric Schwarzschild BH, the line element is given by

$$ds^2 = - \left( 1 - \frac{16\pi G_d M}{(d-2)\Omega_{d-2} r^{d-3}} \right) dt^2 - \left( 1 - \frac{16\pi G_d M}{(d-2)\Omega_{d-2} r^{d-3}} \right)^{-1} dr^2 - r^2 d\Omega_{d-2}^2. \quad (19)$$

where  $\Omega_{d-2}$  is the metric of the unit sphere  $S^{d-2}$  and  $G_d = G_4 L^{d-4}$  is the  $d$ -dimensional Newton's constant and  $L$  the size of the extra dimensions. The horizon radius  $r_h$  is defined by the vanishing of the component  $g_{00}$  and is given by

$$r_h = \left( \frac{16\pi G_d M}{(d-2)\Omega_{d-2}} \right)^{\frac{1}{d-3}} = \omega_d L_{Pl} m^{\frac{1}{d-3}} \quad (20)$$

with

$$\omega_d = \left( \frac{16\pi}{(d-2)\Omega_{d-2}} \right)^{\frac{1}{d-3}}, \quad m = \frac{M}{M_{Pl}}, \quad (21)$$

and  $M_{Pl} = (G_d)^{-\frac{1}{d-3}}$  is the fundamental  $d$ -dimensional Planck mass. From Eq.(20) we observe that the horizon radius increases with the space-time dimension, reflecting the strong gravity effects at small distances.

In the standard case, the Hawking temperature and entropy of a BH of large mass  $M$  are given by [26]

$$T_H = \frac{d-3}{4\pi\omega_d m^{1/(d-3)}} M_{Pl}, \quad S = \frac{d-3}{4L_{Pl}^2} A, \quad (22)$$

where  $A = \Omega_{d-2} r_h^{d-2}$  is the BH horizon area.

Let us then examine the effects to all orders in the Planck length brought by the GUP defined by (10), on the Hawking temperature and entropy. Following the heuristic argument of [53], based on the uncertainty principle, we have

$$T_H = \frac{(d-3)\delta P}{2\pi}. \quad (23)$$

In our framework, we use (10) and consider BH near geometry for which  $\delta X \simeq r_h = \omega_d L_{Pl} m^{\frac{1}{d-3}}$ . In this case, the existence of the minimal length leads to the following non zero BH minimum mass

$$m_0 = \left( \frac{(\delta X)_0}{\omega_d L_{Pl}} \right)^{d-3} = \left( \sqrt{\frac{e}{2}} \frac{\alpha}{\omega_d} \right)^{d-3}. \quad (24)$$

It is interesting to note that the BH minimum mass presents a maximum for  $d = 12$ , and tends asymptotically to zero for  $d \geq 30$ . In terms of the BH mass, the GUP-corrected Hawking temperature is

$$T_H = \frac{(d-3)T_{Pl}}{4\pi\omega_d m^{\frac{1}{d-3}}} \exp \left( -\frac{1}{2} W \left( -\frac{1}{e} \left( \frac{m_0}{m} \right)^{2/(d-3)} \right) \right). \quad (25)$$

From this expression we observe, that the BH temperature is only defined for  $m \geq m_0$ . For a BH with a mass equal to  $m_0$ , the Hawking temperature reaches a maximum given by

$$T_H^{\max} = \frac{(d-3)}{2\sqrt{2}\pi\alpha} T_{Pl}. \quad (26)$$

The corrections to the standard Hawking temperature are obtained by expanding (25) in terms of  $\frac{1}{e} (m_0/m)^{2/(d-3)}$

$$T_H \approx \frac{(d-3)T_{Pl}}{4\pi\omega_d m^{\frac{1}{d-3}}} \left[ 1 + \frac{1}{2e} \left( \frac{m_0}{m} \right)^{2/(d-3)} + \frac{5}{8e^2} \left( \frac{m_0}{m} \right)^{4/(d-3)} + \dots \right]. \quad (27)$$

In the limit  $m_0 = 0$ , the standard expression is recovered. However, as mentioned above  $m_0 \rightarrow 0$  for larger  $d$ , and in this case the standard expression of the BH temperature is also reproduced. This means that BHs with GUP in higher dimensional space-time evaporate completely exactly like in the semiclassical picture.

In figure 1, we show the variation of the corrected Hawking temperature (25) with the BH mass. We observe that BHs in a scenario with extra dimensions are hotter and consequently tends to evaporate faster.

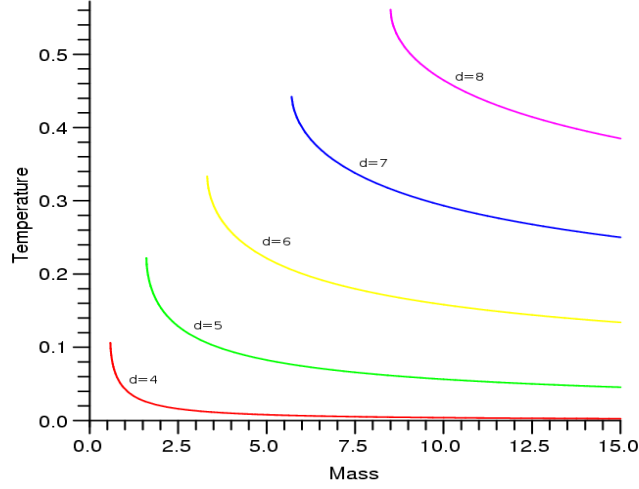


Figure 1: Corrected Hawking temperature versus the black hole mass (in units with  $G = L = 1$ ).

We turn now to the calculation of the micro-canonical entropy of a large BH. Following heuristic considerations due to Bekenstein, the minimum increase of the area of a BH absorbing a classical particle of energy  $E$  and size  $R$  is given by  $(\Delta A)_{min} \simeq \frac{4(\ln 2)L_{Pl}^{d-2}ER}{(d-3)}$ . At the quantum mechanical level, the size and the energy of the particle are constrained to verify  $R \sim 2\delta X$  and  $E \sim \delta P$ . Then we have  $(\Delta A)_{min} \simeq \frac{8(\ln 2)L_{Pl}^{d-2}\delta X\delta P}{(d-3)}$ . Extending this approach to the case with GUP we obtain

$$(\Delta A)_{min} \approx \frac{4(\ln 2)L_{Pl}^{d-2}}{(d-3)} \exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{A_0}{A}\right)^{2/(d-2)}\right)\right), \quad (28)$$

where  $A = \Omega_{d-2}(r_h)^{d-2}$  and  $A_0 = \Omega_{d-2}(\delta X)_0^{d-2}$  are respectively the BH horizon area and minimum horizon area. Considering near horizon geometry, for which we have  $\delta X$  as the horizon radius, and with the aid of the Bekenstein calibration factor for the minimum increase of entropy  $(\Delta S)_{min} = \ln 2$ , we have

$$\frac{dS}{dA} \simeq \frac{5(\Delta S)_{min}}{(\Delta A)_{min}} = \frac{(d-3)}{4L_{Pl}^{d-2}} \exp\left(\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{A_0}{A}\right)^{2/(d-2)}\right)\right). \quad (29)$$

Then, up to an irrelevant constant, we write the entropy as

$$S_d = \frac{(d-3)}{4L_{Pl}^{d-2}} \int_{A_0}^A \exp\left(\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{A_0}{A}\right)^{2/(d-2)}\right)\right) dA. \quad (30)$$

The lower limit of integration is a consequence of the GUP. Using the variable  $y = \frac{1}{e}\left(\frac{A_0}{A}\right)^{2/(d-2)}$  and the relation  $e^{\frac{W(x)}{2}} = \sqrt{x/W(x)}$  we have

$$S_d = -\frac{(d-3)(d-2)}{8(\sqrt{e}L_{Pl})^{d-2}} A_0 \int_{\frac{1}{e}}^{\frac{1}{e}(\frac{A_0}{A})^{2/(d-2)}} y^{-\frac{d}{2}} \left[\frac{-y}{W(-y)}\right]^{\frac{1}{2}} dy, \quad (31)$$



Performing the integration we finally obtain the following corrected BH entropy for some values of  $d$ . Up to a constant, which is the value of the entropy for  $y = 1/e$ , we obtain for  $d = 4$  :

$$S_4 = \frac{A_0}{8(\sqrt{e}L_{Pl})^2} \left[ \frac{2}{\sqrt{zW(z)}} - Ei \left( 1, \frac{1}{2}W(z) \right) \right]_{z=-\frac{1}{e}(A_0/A)} , \quad (32)$$

$d = 5$  :

$$S_5 = - \frac{A_0}{2(\sqrt{e}L_{Pl})^3 z \sqrt{-W(z)}} \left[ 1 + W(z) + \sqrt{\pi} z \sqrt{W(z)} \operatorname{erf} \left( \sqrt{W(z)} \right) \right]_{z=-\frac{1}{e}(A_0/A)^{2/3}} , \quad (33)$$

$d = 6$  :

$$S_6 = \frac{3A_0}{8(\sqrt{e}L_{Pl})^4} \left[ - \frac{2}{z \sqrt{zW(z)}} - \frac{3}{2} Ei \left( 1, \frac{3}{2}W(z) \right) - \frac{1}{z} \sqrt{\frac{W(z)}{z}} \right]_{z=-\frac{1}{e}(A_0/A)^{1/2}} , \quad (34)$$

$d = 7$  :

$$S_7 = \frac{A_0}{3(\sqrt{e}L_{Pl})^5 z^2 \sqrt{-W(z)}} \left[ W(z) - 4(W(z))^2 - 4\sqrt{2\pi} z^2 \sqrt{W(z)} \operatorname{erf} \left( \sqrt{2W(z)} \right) + 3 \right]_{z=-\frac{1}{e}(A_0/A)^{2/5}} , \quad (35)$$

$d = 8$  :

$$S_8 = \frac{5A_0}{64(\sqrt{e}L_{Pl})^6} \left[ 10 \left( \frac{W(z)}{z} \right)^{\frac{3}{2}} - \frac{4}{z^2} \sqrt{\frac{W(z)}{z}} - \frac{16}{z^2 \sqrt{zW(z)}} - 25 Ei \left( 1, \frac{5}{2}W(z) \right) \right]_{z=-\frac{1}{e}(A_0/A)^{1/3}} , \quad (36)$$

where  $\operatorname{erf}(z)$  is the error function and  $Ei(n, z)$  is the exponential integral. The corrections to the standard expressions are obtained by applying a Taylor expansion around the parameter  $z$  which is a small one by virtue of the GUP. For  $d = 4$  we obtain

$$S_4 = \frac{A}{4L_{Pl}^2} - \frac{\pi\alpha^2}{4} \ln \frac{A}{A_0} + \frac{\pi\alpha^3}{16e} \left( \frac{A_0}{A} \right) + \frac{25\pi\alpha^2}{192e^2} \left( \frac{A_0}{A} \right)^2 + \frac{343\pi\alpha^2}{2304e^3} \left( \frac{A_0}{A} \right)^3 + \dots , \quad (37)$$

and for  $d = 5, 6, 7, 8$  we have respectively

$$S_5 = \frac{A}{2L_{Pl}^3} - \frac{3}{4\sqrt{2}} \pi^2 \alpha^3 e^{1/2} \left( \frac{A}{A_0} \right)^{1/3} + \frac{9}{16\sqrt{2}} \frac{\pi^2 \alpha^3}{e^{1/2}} \left( \frac{A_0}{A} \right)^{1/3} + \frac{25}{96\sqrt{2}} \frac{\pi^2 \alpha^3}{e^{3/2}} \left( \frac{A_0}{A} \right) + \frac{196}{1280\sqrt{2}} \frac{\pi^2 \alpha^3}{e^{5/2}} \left( \frac{A_0}{A} \right)^{5/3} + \dots , \quad (38)$$

$$\begin{aligned}
 S_6 = & \frac{3A}{4L_{Pl}^4} - \frac{\pi^2 e \alpha^4}{2} \left( \frac{A}{A_0} \right)^{1/2} - \frac{3}{16} \pi^2 \alpha^4 \ln \frac{A}{A_0} + \frac{25}{48} \frac{\pi^2 \alpha^4}{e} \left( \frac{A_0}{A} \right)^{1/2} \\
 & + \frac{343}{768} \frac{\pi^2 \alpha^4}{e^2} \left( \frac{A_0}{A} \right) + \frac{727}{1280} \frac{\pi^2 \alpha^4}{e^3} \left( \frac{A_0}{A} \right)^{3/2} + \dots,
 \end{aligned} \tag{39}$$

$$\begin{aligned}
 S_7 = & \frac{A}{L_{Pl}^5} - \frac{5}{24\sqrt{2}} \pi^3 \alpha^5 e^{3/2} \left( \frac{A}{A_0} \right)^{3/5} - \frac{15}{32\sqrt{2}} \pi^3 \alpha^5 e^{1/2} \left( \frac{A}{A_0} \right)^{1/5} \\
 & + \frac{125}{192\sqrt{2}} \frac{\pi^3 \alpha^5}{e^{1/2}} \left( \frac{A_0}{A} \right)^{1/5} + \frac{1715}{4608\sqrt{2}} \frac{\pi^3 \alpha^5}{e^{3/2}} \left( \frac{A_0}{A} \right)^{3/5} + \dots,
 \end{aligned} \tag{40}$$

$$\begin{aligned}
 S_8 = & \frac{5A}{4L_{Pl}^6} - \frac{1}{8} \pi^3 \alpha^6 e^2 \left( \frac{A}{A_0} \right)^{2/3} - \frac{3}{16} \pi^3 \alpha^6 e \left( \frac{A}{A_0} \right)^{1/3} \\
 & - \frac{25}{288} \pi^3 \alpha^6 \ln \frac{A}{A_0} + \frac{343}{768} \frac{\pi^3 \alpha^6}{e} \left( \frac{A_0}{A} \right)^{1/3} + \frac{2187}{5120} \frac{\pi^3 \alpha^6}{e^2} \left( \frac{A_0}{A} \right)^{2/3} + \dots
 \end{aligned} \tag{41}$$

We note that we have reproduced in the case with even number of dimensions the log-area correction term with a negative sign, since we are dealing with the micro-canonical entropy. For  $d = 4$ , the expansion coefficients are proportional to  $\alpha^{2(n+1)}$ , exactly as in [49].

In order to analyze the question of how a generalization of the Heisenberg uncertainty principle might influence the BH entropy and then the BH decay, we construct the following ratio between the entropy calculated in different scenarios

$$R_0 = \frac{S_{ao}}{S_H}, \quad R_1 = \frac{S_{ao}}{S_{lo}}, \tag{42}$$

where  $S_{ao}, S_{lo}, S_H$  are respectively the entropy with the GUP to all orders in the Planck length, the entropy with GUP to leading order in the Planck length (see the end of section 4), and the entropy in the Hawking picture. The results of this analysis are shown in Figs. 2 and 3 where we see that  $R_0$  and  $R_1$  increase with  $m$  and decrease with  $d$ . For large  $m$ ,  $R_0$  tends slowly to unity in comparison to  $R_1$ . This shows, that the effects of the GUP become relevant as the mass of the BH decreases.

Regarding our results, we conclude that the BH entropy is smaller than the ones obtained in the Hawking picture and with GUP to leading order in the Planck length. On the other hand the BH entropy decreases with the number of dimensions confirming the predictions of GUP to leading order in the Planck length. This indicates that BHs in scenario with extra dimensions and GUP to all orders in the Planck length have less degrees of freedom compared to their counterparts in the Hawking picture and GUP to leading order in the Planck length. Then, in our framework we expect a significant suppression of the multiplicity of the emitted particles in the evaporation phase.

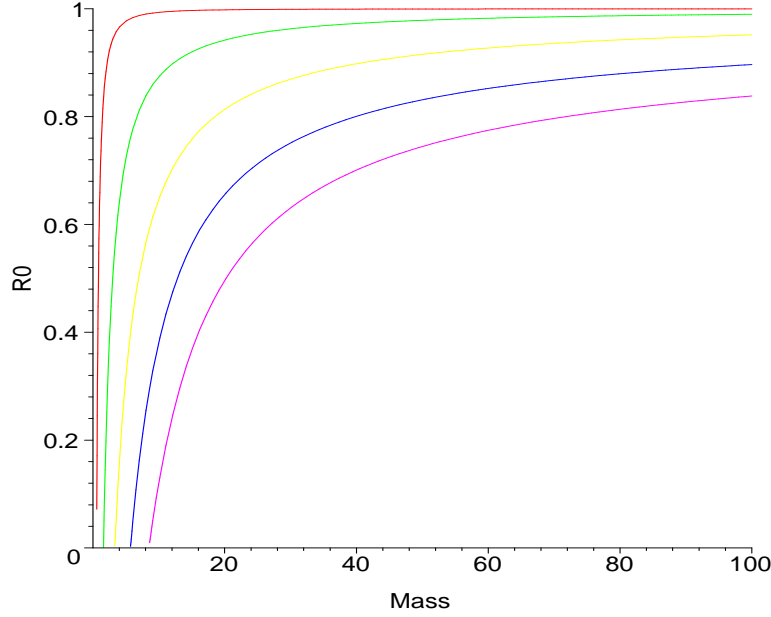


Figure 2: Ratio of the entropy with GUP to all orders in the Planck length to the entropy in the standard Hawking picture as a function of the BH hole mass (in units with  $G = L = 1$ ). From left to right,  $d = 4, 5, 6, 7, 8$ .

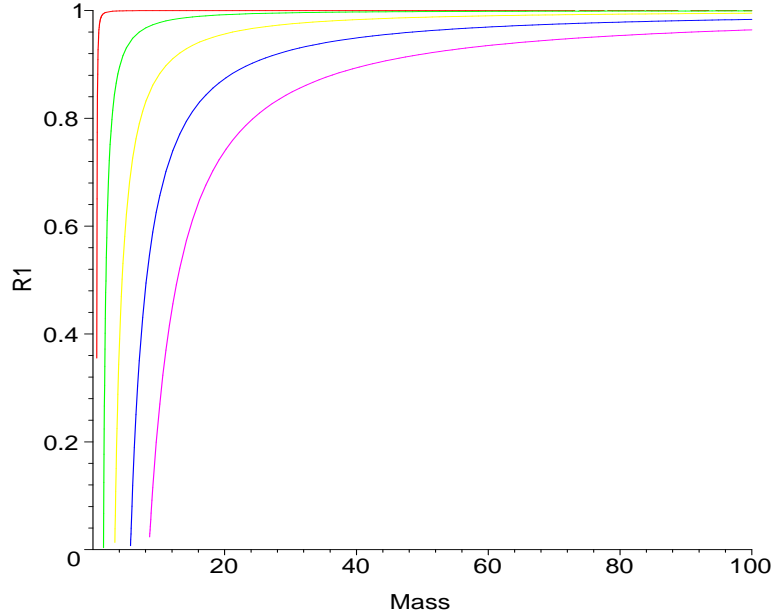


Figure 3: Ratio of the entropy with GUP to all orders in the Planck length to the entropy with GUP to leading order in the Planck length as a function of the BH hole mass (in units with  $G = L = 1$ ). From left to right,  $d = 4, 5, 6, 7, 8$ .

#### 4. Black hole evaporation

We consider now the mass loss rates and lifetimes of a BH of large mass  $m$ . Once produced, the BH undergoes a number of phases before completely evaporating or

leaving an inert BH remnant in the scenario with GUP. These phases are summarized in the following [11]

*Balding phase:* During this phase, the BH lost hair associated with multipole moments inherited from the initial particles, and a fraction of the initial mass will be lost by gravitational radiation.

*Evaporation phase:* The BH starts losing its angular momentum through the emission of Hawking radiation and possibly, through super-radiance and undergoes emission of thermally distributed quanta until the BH reaches the Planck scale. The emitted spectrum contains all Standard model particles, which are emitted on the brane, as well as gravitons, which are also emitted in the bulk direction.

*Planck phase:* During this phase, the semi classical picture breaks down since the mass and/or the Hawking temperature approach the Planck scale. Hence, a theory of quantum gravity is necessary to study this phase. However, it is suggested that the BH will decay to a few quanta with Planck-scale energies or to a inert remnant.

The usual thermodynamical description of the Hawking evaporation process is usually performed with the canonical ensemble (CE) approach. In the CE approach the energies of the emitted particles are small compared to the BH mass. However, it was pointed out in [84], that the CE approach is no longer appropriate in the final stage of evaporation where BH is hot and its mass approaches the Planck scale. Thus, the correct description of the evaporation process requires the use of the micro-canonical ensemble (MCE) description.

In the following, ignoring the contribution of the grey-body factors, we calculate the evaporation rate of a massive BH such that  $m/m_0$  is much greater than  $O(1)$ , where  $m_0$  is the minimum BH mass allowed by the GUP. In this approximation, the MCE corrections can be neglected and the energy density of the emitted particles in (D+1)-dimensional space-time is given by

$$\mathcal{E} = 2\Omega_{D-1} \int_0^\infty \frac{p^D e^{-\alpha^2 L_{Pl}^2 p^2}}{e^{\beta p} - 1} dp. \quad (43)$$

The evaluation of this integral proceeds by expanding the exponential and the use of the following definition of the Riemann Zeta function

$$\int_0^\infty \frac{y^{s-1}}{e^y - 1} dy = \Gamma(s) \zeta(s). \quad (44)$$

As a result we obtain

$$\mathcal{E} = 2\Omega_{D-1} T^{D+1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} (\alpha L_{Pl} T_H)^{2n} \Gamma(2n + D + 1) \zeta(2n + D + 1). \quad (45)$$

The series in Eq.(45) is an alternating series which converge when  $T_H < \alpha^{-1} T_{Pl}$ . However the existence of a maximum value of the Hawking temperature implies a stronger condition on  $T$ . Using the expression of the Hawking maximum temperature given by Eq.(26), we have

$$\alpha \frac{T}{T_{Pl}} < \frac{d-3}{2\pi\sqrt{2}}. \quad (46)$$

This constraint allows us to cut the series at  $n = 1$ . Then we have

$$\mathcal{E} = 2\Omega_{D-1}T^{D+1}\Gamma(D+1)\zeta(D+1) \times \left(1 - \frac{(d-3)^2(D+1)(D+2)\zeta(D+3)}{8\pi^2\zeta(D+1)} \left(\frac{T_H}{T_H^{max}}\right)^2\right). \quad (47)$$

Neglecting thermal emission in the bulk and assuming a  $(D+1)$ -dimensional brane, the intensity emitted by a massless scalar particle on the brane is

$$\frac{dM}{dt} = -A_{D+1}\mathcal{E}(T_H). \quad (48)$$

where  $A_D = \Omega_{D-2}r_c^{D-2}$  is the horizon area of the induced BH and  $r_c = \left[\frac{d-1}{2}\right]^{1/(d-3)} \left[\frac{d-1}{d-3}\right]^{1/2} r_h$  is the critical radius of the BH considered as an absorber [9]. In Eq.(48), the constancy of the surface gravity over the horizon, allows to identify the Hawking temperature of the higher dimensional BH as the temperature of the induced BH on the brane.

Considering a four dimensional brane and using the corrected Hawking temperature given by (25) we obtain

$$\frac{dm}{dt} = -\gamma_1 Z e^{-2W(-Z)} (1 - \gamma_2 \alpha^2 Z e^{-W(-Z)}), \quad (49)$$

with  $\gamma_1 = \frac{\pi^2 e^2 M_{Pl}}{120 \omega_d^2} (r_c/r_h)^2 (d-3)^4 m_0^{-2/(d-3)}$ ,  $\gamma_2 = \frac{5e}{42 \omega_d^2} (d-3)^2 m_0^{-2/(d-3)}$  and  $Z = \frac{1}{e} (m_0/m)^{2/(d-3)}$ .

In figure 4, we show the variation of the evaporation rate with the BH mass. We observe that the evaporation phase ends when the BH mass reaches the minimum mass  $m_0$ . In this case  $Z = 1/e$  and the evaporation rate given by

$$\left(\frac{dm}{dt}\right)_{min} = -\gamma_1 e (1 - \gamma_2 \alpha^2), \quad (50)$$

is finite. It is important to note, that although we have used the CE approach, the usual divergence at the end of the Hawking evaporation in the standard description, is now completely removed by the GUP. However, as pointed by several authors, the divergence at the end of the Hawking evaporation process in the standard description is a consequence of the incorrect use of the CE approach and can be cured by the MCE treatment [15, 84]. In the framework with GUP, the existence of a maximum temperature given by (26) suppress the evaporation process beyond the Planck temperature. This behavior is similar to the prevention, by the standard uncertainty principle, of the hydrogen atom from total collapse.

Performing a Taylor expansion around the small parameter  $Z$  we obtain

$$\frac{dm}{dt} = -\frac{\gamma_1}{e} (1 - \gamma_2 \alpha^2) \left(\frac{m_0}{m}\right)^{\frac{2}{d-3}} - \frac{2\gamma_1}{e^2} \left(1 - \frac{3}{2}\gamma_2 \alpha^2\right) \left(\frac{m_0}{m}\right)^{\frac{4}{d-3}} + \dots, \quad (51)$$

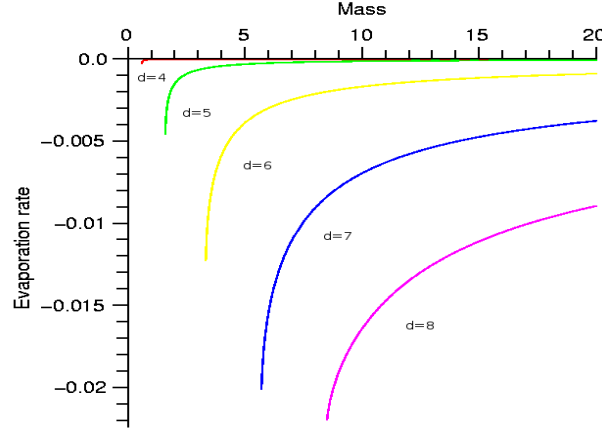


Figure 4: The evaporation rate as a function of the black hole mass (in units with  $G = L = 1$ ).

In the framework with GUP, the Hawking evaporation process of BHs with mass  $m > m_0$  continue until the horizon radius becomes  $(\delta X)_{min}$  leaving a Planck sized BH remnant. The nature of this BH remnant (BHR) is best described by the specific heat. Using the definition  $C_d = \frac{dM}{dT_H}$  we obtain

$$C_d = C_d^0 \left( 1 + W \left( -\frac{1}{e} \left( \frac{m_0}{m} \right)^{2/(d-3)} \right) \right) \exp \left( \frac{1}{2} W \left( -\frac{1}{e} \left( \frac{m_0}{m} \right)^{2/(d-3)} \right) \right), \quad (52)$$

where  $C_d^0 = -4\pi\omega_d L_{Pl} m^{\frac{d-2}{d-3}}$  is the heat capacity without GUP. We observe that the heat capacity vanishes when  $\left( 1 + W \left( -\frac{1}{e} \left( \frac{m_0}{m} \right)^{2/(d-3)} \right) \right) = 0$ , whose solution is given by  $m = m_0$ , corresponding to the end point of the evaporation phase. This state, characterized by a maximal temperature, can be considered as the ground state of the BH. This interpretation is motivated by the fact that the ground state is independent of the temperature [85]. Thus, the vanishing of the specific heat and the entropy at the end of the evaporation reveals, beside gravitational interaction with the surrounding, the inert character of the BHRs and thus make them as potential candidates for the origin of dark matter [54, 55]. We note that, as it is the case with the GUP to leading order in the Planck length widely used in the literature, the BHRs seems to be also consequence of the GUP to all orders in the Planck length considered in this paper [53, 26]. We note that BHRs can be found in different contexts like noncommutative geometry [74, 77] and effective models based on the *Limiting Curvature Hypothesis* (LCE) [86]. However in these scenarios the BH radiates eternally. These models and others have in common that the temperature of the BH reaches a maximum before dropping to zero. In our framework, this behavior is forbidden by the cut-off implemented by the GUP. In figure 5, we show the variation of the specific heat with the BH mass.

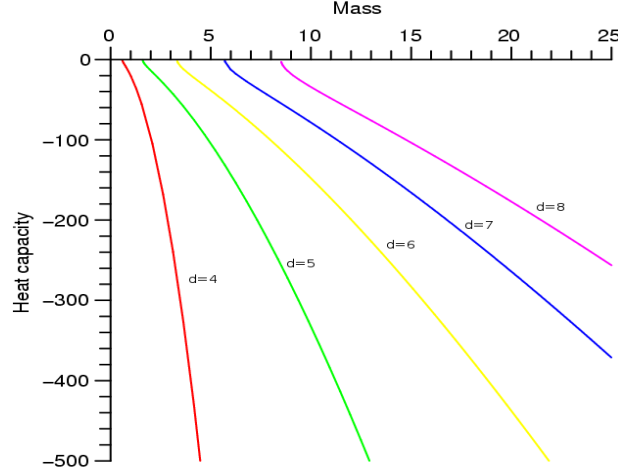


Figure 5: Heat capacity as a function of the black hole mass (in units with  $G = L = 1$ ).

A Taylor expansion around  $\frac{1}{e}(m_0/m)^{2/(d-3)}$  gives

$$C_d = C_d^0 \left[ 1 - \frac{3}{2e} \left( \frac{m_0}{m} \right)^{\frac{2}{d-3}} - \frac{7}{8e^2} \left( \frac{m_0}{m} \right)^{\frac{4}{d-3}} - \frac{55}{6e^3} \left( \frac{m_0}{m} \right)^{\frac{6}{d-3}} \right]. \quad (53)$$

For a BH with a mass larger than the minimum mass allowed by the GUP, the heat capacity can be approximated by the standard expression,  $C_d^0$ . The corrections terms to the specific heat due to GUP are all positive showing that the evaporation process is accelerated, leading to a GUP corrected decay time smaller than the decay time in the standard case.

Taking into account that the evaporation phase ends when the BH mass reaches  $m_0$ , we obtain from (49), the following expression for the decay time

$$t_d = -\frac{(d-3)m_0}{2\gamma_1 e^{(d-3)/2}} \left( I(d+1, d-3, Z_i) + \gamma_2 \alpha^2 I(d-1, d-3, Z_i) \right), \quad (54)$$

where  $Z_i = -\frac{1}{e}(m_0/m_i)^{\frac{2}{d-3}}$ ,  $m_i$  the initial mass of the BH and

$$I(p, q, Z_i) = -(-1)^{\frac{q}{2}} \int_{Z_i}^{-\frac{1}{e}} W(Z)^{-\frac{p}{2}} e^{-\frac{q}{2}W(Z)} dZ, \quad (55)$$

These integrals can be evaluated analytically in terms of the Lambert Function  $W(x)$  and the Whittaker function  $M(\alpha; \beta; x)$ .

A plot of the decay time as a function of the BH mass obtained from a exact evaluation of the integrals in Eq.(55) is shown in Fig. 5. We observe that the decay time is a rapidly decreasing function of the space-time dimension. This confirm the fact that BHs in higher dimensional space-times are hotter and decay faster.

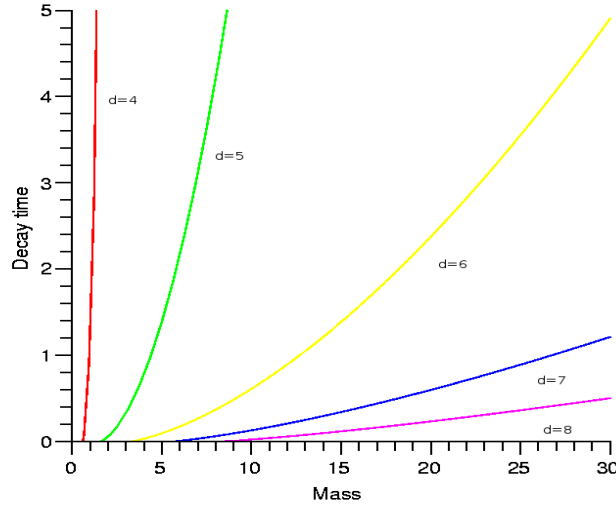


Figure 6: Decay time as a function of the black hole mass (in units with  $G = L = 1$ ).

In the rest of this section, we proceed to a comparison of our results with the ones obtained in the framework of the GUP to leading order in the Planck length [65, 26]. From the saturate GUP defined by Eq.(17) we obtain

$$(\delta P) = \frac{\delta X}{\alpha^2 L_{Pl}^2} \left( 1 - \sqrt{1 - \frac{\alpha^2 L_{Pl}^2}{(\delta X)^2}} \right). \quad (56)$$

This leads to the minimal length given by Eq.(18). Following the same calculations as above, we obtain

$$T_H = \frac{d-3}{2\pi\alpha} T_{Pl} Z^{-1} \left( 1 - \sqrt{1 - Z^2} \right), \quad (57)$$

$$C_d = 2\pi\alpha m_0 Z^{4-d} \frac{\sqrt{1 - Z^2}}{\sqrt{1 - Z^2} - 1}. \quad (58)$$

with  $Z = (m_0/m)^{1/(d-3)}$  and  $m_0 = (\alpha/\omega_d)^{d-3}$  is the minimum BH mass allowed by the GUP. Substituting  $Z = 1$  in Eq.(57), we obtain the maximum BH temperature  $T_H^{max} = \frac{d-3}{2\pi\alpha} T_{Pl}$ .

The calculation of the entropy gives

$$S_d = -\frac{(d-2)(d-3)}{16L_{Pl}^{d-2}} A_0 \int_1^{(A_0/A)^{2/(d-2)}} \frac{y^{\frac{2-d}{2}}}{1 - \sqrt{1-y}} dy, \quad (59)$$

where  $y = (A_0/A)^{2/(d-2)}$ . The integral can be evaluated for given values of  $d$  and we obtain for  $d = 4, 5, 6$  the following expressions

$$S_4 = -\frac{A_0}{8L_{Pl}^2} \left[ \frac{1}{2} \ln \frac{\sqrt{1-y}+1}{\sqrt{1-y}-1} + \frac{1}{\sqrt{1-y}-1} \right]_{y=A_0/A}, \quad (60)$$

$$S_5 = -\frac{A_0}{8L_{Pl}^3} \sqrt{\frac{1-y}{y}} \left[ 1 - \frac{1}{y} - \frac{1}{y\sqrt{1-y}} \right]_{y=(A_0/A)^{2/3}}, \quad (61)$$



$$S_6 = -\frac{A_0}{16L_{Pl}^2} \left[ \frac{1}{2} \ln \frac{\sqrt{1-y}+1}{\sqrt{1-y}-1} - \frac{1}{\sqrt{1-y}+1} - \frac{1}{(\sqrt{1-y}-1)^2} \right]_{y=(A_0/A)^{1/2}}. \quad (62)$$

A Taylor expansion in the parameter  $y$  gives again the log area correction in the case of even dimension.

The calculation of the evaporation rate in the framework with GUP to leading order in the Planck length requires a careful analysis. The calculations done until now used the usual momentum measure in the derivation of the Stefan-Boltzmann law. However, it is easy to show, following [45], that the fundamental cell in the momentum space is squeezed by the presence of the minimal and becomes  $dp/(1 + \alpha^2 L_{Pl}^2 p^2)$ , like the momentum measure given by Eq.(2) in our framework. Then, using the Bose-Einstein statistic, the energy density of the emitted particles is given in the CE approach by

$$\mathcal{E} = 2\Omega_{D-1} \int_0^\infty \frac{p^D}{(1 + \alpha^2 L_{Pl}^2 p^2) e^{\beta p} - 1} dp. \quad (63)$$

We then arrive to the same leading contribution given by Eq.(47), with the condition on the Hawking temperature given now by

$$\frac{T_H}{T_{Pl}} < \frac{d-3}{2\pi}. \quad (64)$$

Using the definition of the evaporation rate given by Eq.(48) we obtain

$$\frac{dm}{dt} = -\alpha_1 Z^{-3} \left(1 - \sqrt{1-Z}\right)^4 \left[1 - \alpha_2 Z^{-1} \left(1 - \sqrt{1-Z}\right)^2\right], \quad (65)$$

where  $Z = (m_0/m)^{2/(d-3)}$ ,  $\alpha_1 = \frac{2^{(d-5)/(d-3)} \pi^2}{15\alpha^4} M_{Pl} \omega_d^2 m_0^{2/(d-3)} (d-3)^3 (d-1)^{\frac{d-1}{d-3}}$ ,  $\alpha_2 = \frac{10}{21} (d-3)^2$ .

The Hawking evaporation ends when the BH mass becomes equal to  $m_0$  with a minimum evaporation rate given by

$$\left(\frac{dm}{dt}\right)_{min} = -\alpha_1 (1 - \alpha_2). \quad (66)$$

The expression of the decay time follows from Eq.(65) and is given by

$$t = \frac{d-3}{2\alpha_1} m_0 \left( I\left(-4, \frac{7-d}{2}, Z_i\right) + \alpha_2 I\left(-2, \frac{5-d}{2}, Z_i\right) \right), \quad (67)$$

where

$$I(p, q, Z_i) = \int_{Z_i}^1 dZ Z^q \left(1 - \sqrt{1-Z}\right)^p. \quad (68)$$

The evaluation of the integrals gives complicated and long expressions to be presented here. Our results differ from the ones obtained in [26, 65] by the presence of the seconds terms in Eqs.(65,67), which are a consequence of the squeezing of the momentum space.

Before ending this section we consider the multiplicity of particles emitted during the evaporation process. Assuming that the BH radiates mainly on the 3-brane, and ignoring the grey body factors the multiplicity is given by [87]

$$N = \frac{d-3}{d-2} \frac{\zeta(3)}{3\zeta(4)} \frac{M}{T_H}, \quad (69)$$

In the case of the GUP to all orders in the Planck length and with the BH temperature given by (25), we observe that, compared to the standard picture, the additional exponential factor leads to a reduction of the average multiplicity with increasing number of extra dimensions.

In table 1, we show the corrected thermodynamics parameters of two five dimensional BHs with initial mass equal to  $5M_{Pl}$  and  $10M_{Pl}$  in the frameworks on the GUP to leading order ( $GUP^{lo}$ ) and the GUP to all orders in the Planck length ( $GUP^{ao}$ ). The first row represents the semiclassical results obtained with the Heisenberg uncertainty principle (HUP). We observe a reduction of the entropy and the decay time in our framework. This reduction becomes significant as the number of extra dimensions increases. As a result, the multiplicity of emitted particles in BH decay is then significantly suppressed. For example, for  $d = 8$  and  $M = 9M_{Pl}$  the multiplicity with  $GUP^{ao}$  is suppressed by a factor  $-12\%$  in comparison to the multiplicity with  $GUP^{fo}$  and by  $-33\%$  in comparison to the standard multiplicity. For  $d = 8$  and  $M = 12M_{Pl}$ , the suppression factors are respectively  $-5\%$  and  $-24\%$ . The reduction of the entropy indicates the breakdown of the semiclassical picture and that BH in the framework with a GUP have less degrees of freedom compared to the standard picture. As it is the case with the  $GUP^{lo}$ , the effects of the GUP becomes stronger as the minimum BH mass increases. These results may have important consequences on possible BHRs production in particle colliders and in ultrahigh energy cosmic ray (UHECR) airshowers.

Table 1. Corrected thermodynamic parameters for two five-dimensional BHs with mass 5 and 10 (in Planck units). The deviations from the results with  $GUP^{lo}$  are also given.

M=5						
	$m_0$	$T_i$	$T_f$	$S$	$t$	$N$
<i>HUP</i>	—	0.077	$\infty$	86.29	2.01	12
<i>GUP<sup>lo</sup></i>	1.178	0.082	0.318	33.52 (-61%)	1.47 (-27%)	11.22 (-6.3%)
<i>GUP<sup>ao</sup></i>	1.60	0.082	0.22	15.56 (-82%)	0.51 (-75%)	11.20 (-6.5%)

M=10						
	$m_0$	$T_i$	$T_f$	$S$	$t$	$N$
<i>HUP</i>	—	0.054	$\infty$	244.07	8.06	33.88
<i>GUP<sup>lo</sup></i>	1.178	0.056	0.318	109.11 (-55%)	7.007 (-13%)	32.85 (-3%)
<i>GUP<sup>ao</sup></i>	1.60	0.056	0.22	53.32 (-78%)	2.53 (-69%)	32.83 (-3%)

## 5. Conclusion

We have considered, in the scenario with large extra dimensions and with a GUP to all orders in the Planck length, the corrections to the BHs thermodynamic parameters. We have obtained exact expressions for the Hawking temperature and entropy. We have also reproduced the log-area logarithmic corrections of the entropy in the case of even number of extra dimensions. Using the canonical ensemble description, we have

investigated the Hawking evaporation process of a large BH and shown that the end of the process is a black hole remnant (BHR) with zero entropy, zero heat capacity and a finite non zero temperature. We have shown that BHs in the framework of a GUP to all orders in the Planck length have less degrees of freedom, are hotter and decay faster than in the Hawking semiclassical picture and in the framework of the GUP to leading order in the Planck length.

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